MATH 5061 Lecture 7 (Mar 3)

[Problem Set 4 posted, due on Mar 17.]

<u>Recall</u>: $(M^n, \overset{\langle , \rangle}{\overset{\shortparallel}{g}}) \rightsquigarrow$ Connection ∇ $\rightsquigarrow$ Curvature R

$$R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

$$\leftrightsquigarrow \quad \underbrace{R(X,Y,Z,W) := \langle R(X,Y)Z, W \rangle}_{(0,4)\text{-tensor}} \qquad \forall X,Y,Z,W \in \Gamma(TM)$$

In local coord. of M, say $(x^1, \dots, x^n)$, let $\partial_i := \frac{\partial}{\partial x^i}$

$$g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle = \langle \partial_i, \partial_j \rangle$$

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} (\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$$

Compute $R(\partial_i, \partial_j, \partial_k, \partial_\ell) =: R_{ijk\ell}$

$$\nabla_{\partial_j} \nabla_{\partial_i} \partial_k = \nabla_{\partial_j} (\Gamma_{ik}^\ell \partial_\ell) = (\partial_j \Gamma_{ik}^s) \partial_s + \Gamma_{ik}^\ell \Gamma_{j\ell}^s \partial_s$$

i.e. $$\nabla_{\partial_j} \nabla_{\partial_i} \partial_k = [\partial_j \Gamma_{ik}^s + \Gamma_{ik}^\ell \Gamma_{j\ell}^s] \partial_s$$

Similarly, $$\nabla_{\partial_i} \nabla_{\partial_j} \partial_k = [\partial_i \Gamma_{jk}^s + \Gamma_{jk}^\ell \Gamma_{i\ell}^s] \partial_s$$

and $$\nabla_{[\partial_i, \partial_j]} \partial_k = 0$$

$$\Rightarrow R(\partial_i, \partial_j, \partial_k, \partial_\ell) = g_{s\ell} \left(\partial_j \Gamma_{ik}^s - \partial_i \Gamma_{jk}^s + \Gamma_{ik}^\ell \Gamma_{j\ell}^s + \Gamma_{jk}^\ell \Gamma_{i\ell}^s \right)$$

ie. $$R_{ijk\ell} = g_{s\ell} (\partial \Gamma + \Gamma * \Gamma) = F(g, \partial g, \partial^2 g)$$

Symmetries

$$\begin{cases} R_{ijkl} + R_{jkil} + R_{kijl} = 0 & \text{(Bianchi)} \\ R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij} \end{cases}$$

Q: How is the Riemann curvature tensor R related to the notion of "Gauss curvature" for surfaces in $\mathbb{R}^3$?

A: "sectional curvature"

Fix $p \in M$, and a 2-dim'l subspace $\sigma \subseteq T_pM$

Def$^{\underline{n}}$: Sectional curvature of σ at $p \in M$ is defined as

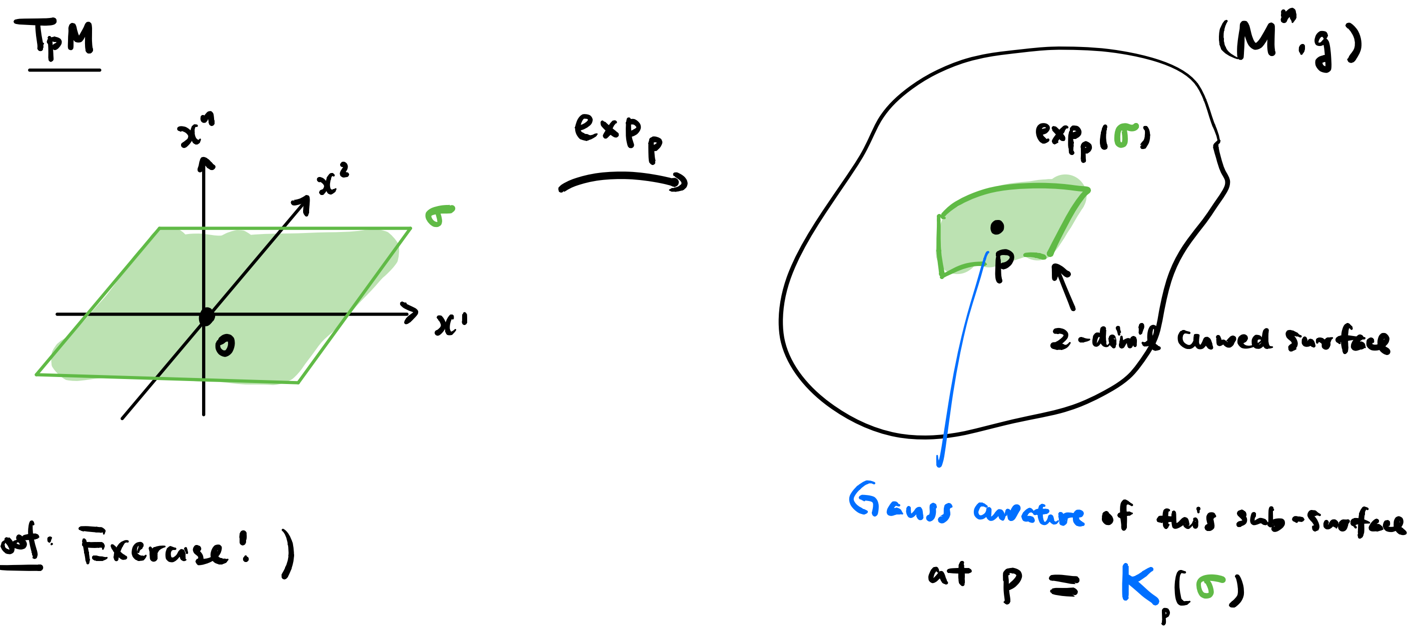
$$K_p(\sigma) := R(e_1, e_2, e_1, e_2)$$

where $\{e_1, e_2\}$ O.N.B. for σ.

FACT: $K(\sigma)$ is "well-defined", ie indep. of the choice of O.N.B $\{e_1, e_2\}$.

Geometric Meaning: $K_p(\sigma) \in \mathbb{R}$ measures the Gauss curvature at p of a "sub-surface" generated by σ in M.

geodesic normal coord.



(Proof: Exercise!)

We have the following algebraic fact.

Prop: Knowing all the sectional curvatures $K_p(\sigma)$ for all $\sigma \subseteq T_pM$ determines completely the Riem. curvature tensor R at p.

Proof: Idea: $R_{ijij} \overset{\text{determines}}{\rightsquigarrow} R_{ijkl}$ using symmetries of R

Let $\{e_1, \dots, e_n\}$ be an O.N.B. for T_pM,

$$\sigma_{ij} := \text{span}\{e_i, e_j\} \subseteq T_pM \quad , \quad i \neq j.$$

$$K(\sigma_{ij}) := R(e_i, e_j, e_i, e_j).$$

Using multi-linearity, only need to know $R(e_i, e_j, e_k, e_l)$, $\begin{matrix} i \neq j \\ k \neq l \end{matrix}$.

Note: $R\left(\frac{e_i + e_k}{\sqrt{2}}, e_j, \frac{e_i + e_k}{\sqrt{2}}, e_j\right) = K\left(\text{span}\left\{\frac{e_i + e_k}{\sqrt{2}}, e_j\right\}\right)$

BUT $R(e_i + e_k, e_j, e_i + e_k, e_j)$

$$= \underbrace{R(e_i, e_j, e_i, e_j)}_{K(\sigma_{ij})} + \underbrace{R(e_k, e_j, e_k, e_j)}_{K(\sigma_{kj})}$$

$$+ R(e_i, e_j, e_k, e_j) + R(e_k, e_j, e_i, e_j)$$

same

$$\Rightarrow \quad R(e_i, e_j, e_k, e_j) = \text{“known”}$$

Note: $R\left(e_i, \frac{e_j + e_l}{\sqrt{2}}, e_k, \frac{e_j + e_l}{\sqrt{2}}\right) = \text{“known”}$

BUT $R(e_i, e_j + e_l, e_k, e_j + e_l)$

$$= \underbrace{R(e_i, e_j, e_k, e_j)}_{\text{“known”}} + \underbrace{R(e_i, e_l, e_k, e_l)}_{\text{“known”}}$$

$$+ R(e_i, e_j, e_k, e_l) + \underbrace{R(e_i, e_l, e_k, e_j)}_{-R(e_j, e_k, e_i, e_l)}$$

i.e. $R(e_i,e_j,e_k,e_l) - R(e_j,e_k,e_i,e_l) = $ "known"

$-)$ $R(e_k,e_i,e_j,e_l) - R(e_i,e_j,e_k,e_l) = $ "known"

$2R(e_i,e_j,e_k,e_l) + R(e_i,e_j,e_k,e_l) = $ "known"

$\square$

Cor: Let $c \in \mathbb{R}$ be a constant.

$K(\sigma) \equiv c$ $\forall \sigma \subseteq T_pM$ $\iff$ $R(X,Y,Z,W) = c\,(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle)$

[Check: $R(e_i,e_j,e_i,e_j) = c\,(\underbrace{\langle e_i,e_i\rangle}_{1}\underbrace{\langle e_j,e_j\rangle}_{1} - \underbrace{\langle e_j,e_i\rangle}_{0}\underbrace{\langle e_i,e_j\rangle}_{0})$]

Ricci and scalar curvature

Let $\{e_1,\dots,e_n\}$ be an O.N.B. for T_pM.

Def^{n}: Ricci curvature $\quad Ric(X,Y) := \sum_{i=1}^{n} R(X,e_i,Y,e_i)$

Scalar curvature $\quad S := \sum_{i=1}^{n} Ric(e_i,e_i)$

FACT: well-defined, indep. of choice of O.N.B. $\{e_1,\dots,e_n\}$.

In local coord.,

$$R_{ijkl} \xrightarrow{\text{"trace"}} R_{ik} := g^{jl} R_{ijkl} \xrightarrow{\text{"trace"}} R := g^{ik} R_{ik}$$

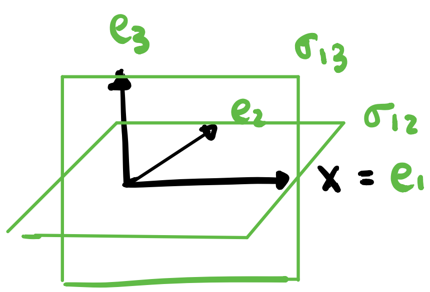
Riem — (0,4)-tensor

Ricci — (0,2)-tensor "symmetric"

Scalar — function

Geometric meaning: Ric & S are "averaged" sectional curvatures:

O.N.B. $\{X = e_1, e_2, \dots, e_n\}$



$$\begin{aligned} Ric(X,X) &= Ric(e_1,e_1) \\ &:= \sum_{i=1}^{n} R(e_1, e_i, e_1, e_i) \\ &= \underbrace{\sum_{i=2}^{n} \overbrace{R(e_1, e_i, e_1, e_i)}^{K(\sigma_{1i})}}_{\text{sum of sect. curv. of planes through } e_1 = X} \end{aligned}$$

Similarly, $S := \sum_{i=1}^{n} Ric(e_i, e_i) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} R(e_i, e_j, e_i, e_j) \right)$

$$= \underbrace{\sum_{i \neq j} R(e_i, e_j, e_i, e_j)}_{\text{sum of all sectional curv.}}$$

A Central Question in Riemannian Geometry

How does the Riem / Ric / Scalar curvatures affect the local / global geometry of (M^n, g)?

E.g.) Gauss-Bonnet Thm: $\iint_S K \, da = 2\pi \chi(S)$.

Now, we digress a bit to talk about covariant derivatives of general tensors ......

Recall: A connection ∇ induces a covariant derivative for vector fields (i.e. $(1,0)$-tensor):

Fix $X \in \mathcal{T}(TM)$.

$$\begin{array}{cccc} \nabla_X : & \mathcal{T}(TM) & \longrightarrow & \mathcal{T}(TM) \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & Y & \longmapsto & \nabla_X Y \end{array}$$

Q: How to covariant differentiate other tensors?

A: "Liebniz rule"

(i.e. (0,1)-tensors)

1-forms: $\omega \in \Omega^1(M) = \mathcal{T}(T^*M) \rightsquigarrow \nabla_X \omega \in \Omega^1(M)$ defined as

$$\left(\underset{\text{v.f.}}{\nabla_X} \underset{\text{1-form}}{\omega}\right)(\underset{\text{v.f.}}{Y}) := \underset{\text{v.f.}}{X}\big(\underbrace{\omega(Y)}_{\text{function}}\big) - \underset{\text{1-form}}{\omega}\big(\underbrace{\nabla_X Y}_{\text{v.f.}}\big)$$

(1,1)-tensors: $\alpha \in \mathcal{T}(T^1_1 M) \rightsquigarrow \nabla_X \alpha \in \mathcal{T}(T^1_1 M)$ defined as

$$(\nabla_X \alpha)(Y, \omega) := X(\alpha(Y,\omega)) - \alpha(\nabla_X Y, \omega) - \alpha(Y, \nabla_X \omega)$$

Example 1: (M^n, g) g: (0,2)-tensor $\rightsquigarrow \exists!$ connection ∇

metric compatibility $\Longleftrightarrow$ $\boxed{\nabla g \equiv 0}$ ie $\nabla_X g \equiv 0 \ \forall X$

why? $$\underbrace{(\nabla_X g)(Y,Z)}_{\substack{\parallel \\ 0}} := \underbrace{X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)}_{\substack{\parallel \\ 0}}$$

$\nabla g \equiv 0$ metric compatibility

Example 2: (Riem. curvature acting on 1-form)

Let $\omega \in \Omega^1(M)$. Define:

$$R(X,Y)\,\omega := \nabla_Y \nabla_X \omega - \nabla_X \nabla_Y \omega + \nabla_{[X,Y]}\omega$$

FACT: $(R(X,Y)\,\omega)(Z) = -\omega(R(X,Y)Z)$

Pf: $(R(X,Y)\omega)(Z)$

$$= (\nabla_Y\nabla_X\omega - \nabla_X\nabla_Y\omega + \nabla_{[X,Y]}\omega)(Z)$$

$$= Y\big((\nabla_X\omega)(Z)\big) - (\nabla_X\omega)(\nabla_Y Z)$$
$$- X\big((\nabla_Y\omega)(Z)\big) + (\nabla_Y\omega)(\nabla_X Z)$$
$$+ [X,Y]\big(\omega(Z)\big) - \omega(\nabla_{[X,Y]}Z)$$

$$= Y\big(X(\omega(Z)) - \omega(\nabla_X Z)\big) - X(\omega(\nabla_Y Z)) + \omega(\nabla_X\nabla_Y Z)$$
$$- X\big(Y(\omega(Z)) - \omega(\nabla_Y Z)\big) + Y(\omega(\nabla_X Z)) - \omega(\nabla_Y\nabla_X Z)$$
$$+ [X,Y]\big(\omega(Z)\big) - \omega(\nabla_{[X,Y]}Z)$$

$$= -\omega(R(X,Y)Z).$$

$\blacksquare$

1st Bianchi identity: $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$

2nd Bianchi identity: $(\nabla_X R)(Y,Z,W,T) + (\nabla_Y R)(Z,X,W,T) + (\nabla_Z R)(X,Y,W,T) = 0$

(Pf: Exercise!)

Submanifold theory

Goal: Generalize the classical theory for surfaces in $\mathbb{R}^3$ to submanifolds $\Sigma^k \subseteq (M^n, g)$

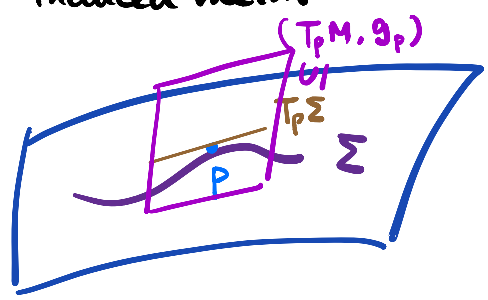
Def$^{\underline{n}}$: An isometric immersion $F : (\Sigma^k, h) \to (M^n, g)$ is an immersion (as manifolds) s.t. $F^* g = h$

Riem. manifold.

As far as local aspects are concerned, we can regard

$\Sigma^k \subseteq (M^n, g)$ and $g|_\Sigma$ induced metric

as a k-dim'l embedded submfd



Note: $(\Sigma^k, g|_\Sigma)$ Riem mfd $\xrightarrow[\text{of R.G.}]{\text{Fund. Thm.}}$ $\exists!$ Riem connection ∇^Σ on Σ

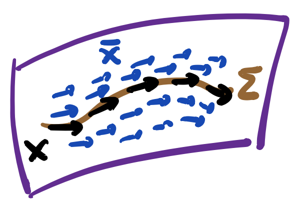
Q: How are the connections ∇^M and ∇^Σ related?

Recall: $\exists$ "canonical" orthogonal splitting: at each $p \in \Sigma$

$$T_pM = T_p\Sigma \oplus \underbrace{(T_p\Sigma)^\perp}_{\text{normal bundle } N_p\Sigma}$$

w.r.t. g

$$\begin{matrix} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v & = & v^T & + & v^N \end{matrix}$$



Thm: Let $X, Y \in \Gamma(T\Sigma)$. Then

$$\nabla^\Sigma_X Y = \left(\nabla^M_{\bar{X}} \bar{Y} \right)^T$$

where $\bar{X}, \bar{Y} \in \Gamma(TM)$ are "extensions" of X, Y s.t. $\bar{X}|_\Sigma = X$, $\bar{Y}|_\Sigma = Y$